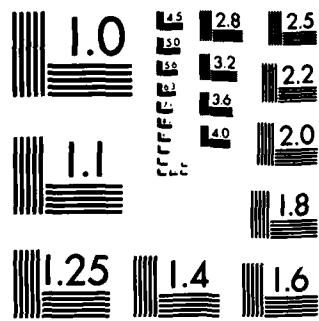


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STOCHASTIC PROPERTIES OF A SEQUENCE
OF INTERFAILURE TIMES
UNDER MINIMAL REPAIR AND UNDER REVIVAL

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Most of the literature on the topic of interfailure times following repair is based on the assumption of maximal repair; that is, a failed item is restored to a condition equal to that of a new item. For some applications, the validity of this assumption has been challenged. In this paper we study the situation wherein the failed item is assumed to have been restored to a condition equal to its condition just prior to failure. This is known as minimal repair. We contrast the two repair policies, cite some implications of minimal repair, and obtain some preservation properties. Finally, we draw attention to the fact that under certain conditions, minimal repair actions generate a class of survival functions which cannot be described by any of the well known properties of ageing studied in reliability theory.

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1. INTRODUCTION

In this paper we attempt to describe the stochastic behavior of a sequence of random variables which arise in a study of repairable systems. Much of the published literature on this topic is based on the assumption that the repair of a failed item restores it to the status of a new item, so that the available machinery of renewal theory can be applied. Such perfect repair actions will be termed "maximal repair." In a series of papers, mostly published in the engineering literature, Ascher, 1968, 1979, and Ascher and Feingold, 1978, 1979, have questioned the reasonableness of maximal repair in many applications, and have thus questioned the value of the ensuing results. Their assertion is that in practice repair is imperfect, in the sense that a failed item is often restored to a condition which is the same as its condition *just prior* to failure, or at best, to a condition which is slightly better. Thus the need for a more realistic description of the stochastic behavior of interfailure times following repair is germane, and this paper is a preliminary effort towards that goal. Here, we shall focus attention on repair actions which restore a failed item to a condition equal to its condition just prior to failure. Barlow and Hunter, 1960, term such imperfect repair actions "minimal repair actions"; Ascher and Feingold, 1969, use the more colloquial expression "bad as old" to refer to the state of the item after repair. It is appropriate to mention here that in Lewis, 1964, and in Brown and Proschan, 1980, certain aspects of imperfect repair have been considered; however, their models, approaches, and the nature of their results are quite different from ours.

Examples of minimal repair actions are:

- (1) A TV set has stopped functioning because of the failure of an integrated circuit (I-C) panel. The set functions as soon as the failed panel is replaced. If the other components are left

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alone, the set is not like a new one; minimal repair has been performed.

(2) A tire which has several miles on it is punctured by a nail and goes flat; the vehicle using the tire is considered to have failed. A repair of the puncture restores the vehicle to an operational status. If we assume that the puncture patch has not strengthened the tire by a significant amount, then a minimal repair has been performed on the vehicle.

(3) A coronary occlusion may cause heart failure. Cardiopulmonary resuscitation (CPR) may revive the patient. Assuming revival without damage to vital organs, we may view CPR as minimal repair.

1.1 Notation and Preliminaries

Let Ω be a measure space endowed with a probability measure P defined on the class L of Borel sets in Ω . Let I denote the set of positive integers, and for every $n \in I$, we shall define a finite, nonnegative, and real valued function $X(n;\omega)$ which is a P -measurable function of $\omega \in \Omega$. In the interest of brevity, we write $X(n)$ instead of $X(n;\omega)$, and note that $\{X(n)\}$ is a stochastic process whose domains are the sets I and Ω .

For our development, we shall require that $X(n) \leq X(n+1)$, $n = 1, 2, \dots$; the motivation for this requirement will be clarified later.

For every fixed n , $X(n)$ is a random variable whose distribution function we shall denote by $F_{X(n)}(x)$. The survival function $P\{X(n) \geq x\}$ is denoted by $\bar{F}_{X(n)}(x) = 1 - F_{X(n)}(x)$.

For an arbitrary finite set of n -values, say $n = 1, 2, \dots, m$, the corresponding random variables $X(1), X(2), \dots, X(m)$ will have a joint m -dimensional distribution, with distribution function

$$F_{X(1), \dots, X(m)}(x_1, \dots, x_m) = P\{X(1) \leq x_1, \dots, X(m) \leq x_m\}.$$

The family of all these joint probability distributions for

$m = 1, 2, \dots$, and all possible values of (x_1, \dots, x_m) , constitutes the family of finite dimensional distributions associated with the $\{X(n)\}$ process. Since I is discrete, the family of finite dimensional distributions uniquely determines the probability that the point $X = (X(1), X(2), \dots)$ belongs to any Borel set of R^∞ .

We shall designate the life length of a new item by $X(1)$, and whenever there is no cause for confusion, write $F(x)$ for $F_{X(1)}(x)$. In what follows we present certain useful notions which are standard in reliability theory.

The failure rate $r(x) \stackrel{\text{def}}{=} \lim_{t \rightarrow 0} (F(x+t) - F(x))/t(1-F(x))$ is assumed to exist, and the cumulative failure rate $R(x) = \int_0^x r(u)du$ is related to the survival function $\bar{F}(x)$, by $\bar{F}(x) = \exp[-R(x)]$.

Since items subjected to repair and other maintenance actions are those which age or experience wearout, a characterization of wear (see Barlow and Proschan, 1975, Ch. 4) given in Definition 1 will be useful. The notation " $H(x) \uparrow(\downarrow) x$ " denotes that the function $H(x)$ is nondecreasing (nonincreasing) in x .

Definition 1. A distribution F (or its survival function \bar{F}) with $F(0) = 0$ is

- (a) IFR(DFR) if $\bar{F}(x+t)/\bar{F}(x) \uparrow(\uparrow) x$, for $x \geq 0$ and each $t \geq 0$;
- (b) IFRA(DFRA) if $-x^{-1}\log\bar{F}(x) \uparrow(\downarrow) x$, for $x > 0$;
- (c) DMRL(IMRL) if $\int_x^\infty \bar{F}(t|x)dt \uparrow(\uparrow) x$, for $x \geq 0$;
- (d) NBU(NWU) if $\bar{F}(x+y) \leqslant (\geqslant) \bar{F}(x)\bar{F}(y)$, for $x, y \geq 0$;
- (e) NBUE(NWUE) if $\int_0^\infty \bar{F}(t+x|x)dt \leqslant (\geqslant) \int_0^\infty \bar{F}(t)dt$, for $x \geq 0$;

$\bar{F}(t|x)$ denotes the conditional reliability of a unit of age x .

The notation " $A \Rightarrow B$ " denotes that A implies B . The following chain of implications is well known (see, for example, Haines and Singpurwalla, 1974, p. 62):

$$\begin{aligned} \text{NBUE} &\Leftarrow \text{DMRL} \Leftarrow \text{IFR} \Rightarrow \text{IFRA} \Rightarrow \text{NBU} \Rightarrow \text{NBUE} \\ \text{NWUE} &\Leftarrow \text{IMRL} \Leftarrow \text{DFR} \Rightarrow \text{DFRA} \Rightarrow \text{NWU} \Rightarrow \text{NWUE}. \end{aligned}$$

2. STOCHASTIC PROCESSES GENERATED BY MINIMAL AND MAXIMAL REPAIR

For convenience, we denote our collection of random variables $\{X(n); X(n+1) \geq X(n), n \in I\}$, where $X(n)$ is the time of the successive failure, by $\{X(n)\}$. Based upon considerations which describe minimal repair, we note

Property 1. The stochastic process $\{X(n)\}$, with $X(0) \stackrel{\text{def}}{=} 0$, generated by a minimal repair policy is such that for all $n \in I$,

$$\begin{aligned} & P\{X(n+1) \geq y \mid X(n) = x, X(n-1) = z, \dots\} \\ &= P\{X(n+1) \geq y \mid X(n) = x\} = P\{X(1) \geq y \mid X(1) \geq x\} \end{aligned}$$

for all $z < x, \dots$, and all $0 \leq x < y < \infty$.

Property 1 states that for each $n \in I$, the distribution of life length following the n th failure, given that the n th failure occurred at x , is the same as the distribution of the first life length given that it was at least x . This characterizes a minimal repair action, for such an action restores the item to the operable state it was in just before failure. In effect, the failure and the associated maintenance action have no discernible effect on the ageing process.

It follows from the above (also see Blumenthal, Greenwood, and Herbach, 1976), that the counting process generated by a minimal repair policy is a nonhomogeneous Poisson process having an intensity rate equal to the failure rate of the life distribution F . Specifically, if $N(x, t]$ denotes the number of minimal repair actions that occur in $(x, t]$, $0 \leq x < t$, then for all $m \in I$,

$$P\{N(x, t] = m\} = e^{-(R(t)-R(x))} (R(t) - R(x))^m / m!.$$

In contrast to minimal repair we have maximal repair, wherein a failed item is either totally overhauled so that it becomes as good as new, or better still, replaced by a new unit. For this we have

Property 2. The stochastic process $\{X(n)\}$, with $X(0) \stackrel{\text{def}}{=} 0$, generated by a maximal repair policy is such that for all $n \in I$,

$$\begin{aligned} & P\{X(n+1) \geq y \mid X(n) = x, X(n-1) = z, \dots\} \\ &= P\{X(n+1) \geq y \mid X(n) = x\} = P\{X(1) \geq y - x\} \end{aligned}$$

for $z < x, \dots$, and all $0 \leq x < y < \infty$.

Note that a maximal repair policy generates an ordinary renewal process. Also, both the minimal and the maximal repair policies exhibit a Markov property.

2.1 A Characterization of the Exponential Distribution

We use the notation of Section 1, and let $\bar{F}_{X(n+1)|X(n)}(y|x)$ denote $P\{X(n+1) \geq y \mid X(n) = x\}$. Then, under a minimal repair policy, $\{X(n)\}$ is such that

$$\bar{F}_{X(n+1)|X(n)}(y|x) = \begin{cases} 1 & , \text{ for } 0 \leq y \leq x \\ \frac{\bar{F}_{X(1)}(y)}{\bar{F}_{X(1)}(x)}, & \text{for } 0 \leq x < y < \infty \end{cases} \quad (2.1)$$

and $\bar{F}_{X(1)}(x) \neq 0$, and under a maximal repair policy $\{X(n)\}$ is such that

$$\bar{F}_{X(n+1)|X(n)}(y|x) = \bar{F}_{X(1)}(y-x), \text{ for } 0 \leq x \leq y < \infty. \quad (2.2)$$

The exponential distribution has played a unique role in reliability theory, by virtue of the fact that in Definition 1, it is the only distribution which is a member of every defining class and its dual. For example, the exponential distribution is both IFR and DFR. An analogous role is played by this distribution with respect to minimal and maximal repair policies. We state this more precisely in the following characterization theorem.

Theorem 1. The stochastic process $\{X(n)\}$ satisfies both (2.1) and (2.2) if and only if $\bar{F}_{X(1)}$ is an exponential survival function.

3. PRESERVATION PROPERTIES OF A MINIMAL REPAIR POLICY

For items subjected to minimal repair, interest often centers around the conditional interfailure time $Y(n+1)$ given $X(n)$, where $\{Y(n+1) = X(n+1) - X(n)\}$ and $X(n)$ is the time of last failure/restoration. Of particular interest are the ageing characteristics of $Y(1) = X(1)$, the time to failure of a fresh unit. Our goal here is to describe those characteristics of $X(1)$ that are preserved by $Y(n+1)|X(n)$ when a minimal repair policy is in effect. Equation (2.1) implies that $\{Y(n)\}$ is such that

$$\bar{F}_{Y(n+1)|X(n)}(y|x) = \bar{F}_{X(n+1)|X(n)}(y+x|x) = \frac{\bar{F}_{X(1)}(y+x)}{\bar{F}_{X(1)}(x)}.$$

In what follows, we adopt the convention that a property, say π , holds for the survival function $\bar{F}_{Y(n+1)|X(n)}(y|x)$, if and only if it holds for all values of x . We introduce

Definition 2. A class of distributions, say S , is said to be *closed under minimal repair*, if $\{X(n)\}$ satisfies (2.1), and

$$\bar{F}_{X(1)} \in S \Rightarrow \bar{F}_{Y(n+1)|X(n)} \in S, \quad \forall n \in I.$$

When the above implication fails to hold, we shall say that S is *not closed under minimal repair*.

Theorem 2, given next, is a key result of this paper. Figure 1 illustrates the implications of the theorem for the IFR class and its associated chain of implications. If an implication is not shown, then there exists an example to show that it is not true.

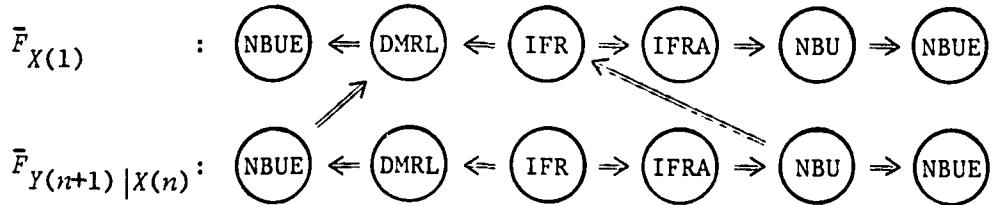


FIGURE 1. Chain indicating which of the ageing characteristics of $\bar{F}_{X(1)}$ are preserved by $\bar{F}_{Y(n+1)|X(n)}$.

Theorem 2.

- (a) The IFR and DMRL classes are closed under minimal repair;
- (b) The IFRA, NBU, and NBUE classes are not closed under minimal repair.

Analogous results hold for the DFR class and its associated chain of implications.

Proof: The proof consists of the series of lemmas given below.

Lemma 1. Let $\{X(n)\}$ satisfy (2.1). Then for all $n \in I$,

- (a) $\bar{F}_{X(1)}^{\text{IFR}(DFR)} \Rightarrow \bar{F}_{Y(n+1)|X(n)}^{\text{IFR}(DFR)}$,
- (b) $\bar{F}_{Y(n+1)|X(n)}^{\text{NBU}(NWU)} \Rightarrow \bar{F}_{X(1)}^{\text{IFR}(DFR)}$,
- (c) $\bar{F}_{Y(n+1)|X(n)}^{\text{NBUE}(NWUE)} \Rightarrow \bar{F}_{X(1)}^{\text{DMRL}(IMRL)}$.

Proof: $X(n)$ satisfying (2.1) \Rightarrow for any $\delta > 0$, and $y \geq 0$,

$$\bar{F}_{Y(n+1)|X(n)}(y+\delta|x) = \frac{\bar{F}_{X(1)}(y+x+\delta)}{\bar{F}_{X(1)}(x)}.$$

Thus

$$\frac{\bar{F}_{Y(n+1)|X(n)}(y+\delta|x)}{\bar{F}_{Y(n+1)|X(n)}(y|x)} = \frac{\bar{F}_{X(1)}(y+x+\delta)}{\bar{F}_{X(1)}(x)} \cdot \frac{\bar{F}_{X(1)}(x)}{\bar{F}_{X(1)}(y+x)}.$$

However,

$$\bar{F}_{X(1)}^{\text{IFR}(DFR)} \Rightarrow \frac{\bar{F}_{X(1)}(y+x+\delta)}{\bar{F}_{X(1)}(y+x)} \downarrow(\uparrow) y,$$

and this establishes part (a) of the lemma. To prove part (b), let us assume that $\bar{F}_{X(1)}$ is not IFR. Then for some x , t , and δ ,

$$\frac{\bar{F}_{X(1)}(x+\delta+t)}{\bar{F}_{X(1)}(x+\delta)} > \frac{\bar{F}_{X(1)}(x+t)}{\bar{F}_{X(1)}(x)}.$$

Rearranging, dividing both sides of the above inequality by $\bar{F}_{X(1)}(x)$, and using the fact that $\{X(n)\}$ satisfies (2.1), we have

$$\bar{F}_{X(n+1)|X(n)}(x+\delta+t|x) > \bar{F}_{X(n+1)|X(n)}(x+t|x)\bar{F}_{X(n+1)|X(n)}(x+\delta|x).$$

Thus $\bar{F}_{Y(n+1)|X(n)}$ is not NBU, and the result follows.

To prove part (c), we note that if $\bar{F}_{Y(n+1)|X(n)}$ is NBUE(NWUE), then for $\tau \geq 0$,

$$\int_0^\infty \frac{\bar{F}_{Y(n+1)|X(n)}(\tau+y|x)}{\bar{F}_{Y(n+1)|X(n)}(y|x)} d\tau \leq (\geq) \int_0^\infty \bar{F}_{Y(n+1)|X(n)}(\tau|x)d\tau$$

So

$$\int_0^\infty \frac{\bar{F}_{X(n+1)|X(n)}(\tau+y+x|x)}{\bar{F}_{X(n+1)|X(n)}(y+x|x)} d\tau \leq (\geq) \int_0^\infty \bar{F}_{X(n+1)|X(n)}(\tau+x|x)d\tau,$$

which implies

$$\int_0^\infty \frac{\bar{F}_{X(1)}(\tau+y+x)}{\bar{F}_{X(1)}(y+x)} d\tau \leq (\geq) \int_0^\infty \frac{\bar{F}_{X(1)}(\tau+x)}{\bar{F}_{X(1)}(x)} d\tau$$

and, hence,

$$\int_{y+x}^\infty \bar{F}_{X(1)}(\tau|y+x)d\tau \leq (\geq) \int_x^\infty \bar{F}_{X(1)}(\tau|x)d\tau,$$

which establishes that $X(1)$ is DMRL(IMRL).

To show closure under minimal repair of the DMRL and the IMRL classes of life distributions, we shall state and prove

Lemma 2. Let $\{X(n)\}$ satisfy (2.1). Then, for all $n \in I$,

$$\bar{F}_{X(1)} \text{DMRL(IMRL)} \Rightarrow \bar{F}_{Y(n+1)|X(n)} \text{DMRL(IMRL)}.$$

Proof: $\bar{F}_{X(1)} \text{DMRL(IMRL)} \Rightarrow$

$$\int_0^\infty \frac{\bar{F}_{X(1)}(t+y+x)}{\bar{F}_{X(1)}(y+x)} dt \leq (\geq) \int_0^\infty \frac{\bar{F}_{X(1)}(t+y+\delta+x)}{\bar{F}_{X(1)}(y+\delta+x)} dt$$

for all $t \geq 0$ and $\delta \geq 0$. But then

$$\begin{aligned} & \int_0^\infty \frac{\bar{F}_{X(1)}(t+y+x)}{\bar{F}_{X(1)}(x)} / \left[\frac{\bar{F}_{X(1)}(y+x)}{\bar{F}_{X(1)}(x)} \right] dt \\ & \leq (\geq) \int_0^\infty \frac{\bar{F}_{X(1)}(t+y+\delta+x)}{\bar{F}_{X(1)}(x)} / \left[\frac{\bar{F}_{X(1)}(y+\delta+x)}{\bar{F}_{X(1)}(x)} \right] dt \end{aligned}$$

and hence,

$$\int_0^\infty \frac{\bar{F}_{Y(n+1)|X(n)}(t+y|x)}{\bar{F}_{Y(n+1)|X(n)}(y|x)} dt \leq (\geq) \int_0^\infty \frac{\bar{F}_{Y(n+1)|X(n)}(t+\delta+y|x)}{\bar{F}_{Y(n+1)|X(n)}(y+\delta|x)}.$$

But this implies that $\bar{F}_{Y(n+1)|X(n)}(\cdot|x)$ is DMRL(IMRL) for each x .

Using Figure 1 and its DFR class counterpart, together with the well known facts that IFRA $\not\Rightarrow$ IFR and DFRA $\not\Rightarrow$ DFR, we can easily establish that the IFRA and DFRA classes are not closed under minimal repair. However, we have the following much stronger result.

Lemma 3. Let $\{X(n)\}$ satisfy (2.1). Then for all $n \in I$, $\bar{F}_{X(1)} \text{IFRA(DFRA)} \not\Rightarrow \bar{F}_{Y(n+1)|X(n)} \text{NBUE(NWUE)}$.

The details of the proof of this lemma are given in Balaban, 1978. For the IFRA class it rests on the following counterexample:

$$\bar{F}_{X(1)}(x) = \begin{cases} 1 & , 0 \leq x < 1 \\ e^{2(1-x)}, & 1 \leq x < 2 \\ e^{-x}, & 2 \leq x. \end{cases}$$

For the DFRA class the following counterexample is used:

$$\bar{F}_{X(1)}(x) = \begin{cases} e^{-x}, & 0 \leq x < 1 \\ e^{-\sqrt{x}}, & 1 \leq x < 4 \\ e^{-x/2}, & 4 \leq x. \end{cases}$$

This lemma has important practical consequences, since it establishes that IFRA(DFRA) life lengths under the action of minimal repair generate conditional distributions whose properties

cannot be described by any of the well known notions of ageing studied in reliability theory. More about this will be said in Section 5.

Nonclosure of the NBU (NWU) and the NBUE (NWUE) classes under minimal repair is proved in Lemma 4, part (a) of which is a more general statement of lack of closure.

Lemma 4. Let $\{X(n)\}$ satisfy (2.1). Then, for all $n \in I$,

- (a) $\bar{F}_{X(1)}^{\text{NBU(NWU)}} \not\Rightarrow \bar{F}_{Y(n+1)|X(n)}^{\text{NBUE(NWUE)}}$,
- (b) $\bar{F}_{X(1)}^{\text{NBUE(NWUE)}} \not\Rightarrow \bar{F}_{Y(n+1)|X(n)}^{\text{NBUE(NWUE)}}$.

Proof: We prove part (a) by contradiction; the proof of part (b) is almost identical to the proof of part (a). Suppose that

$$\bar{F}_{X(1)}^{\text{NBU(NWU)}} \Rightarrow \bar{F}_{Y(n+1)|X(n)}^{\text{NBUE(NWUE)}}.$$

But $\bar{F}_{Y(n+1)|X(n)}^{\text{NBUE(NWUE)}} \Rightarrow \bar{F}_{X(1)}^{\text{DMRL(IMRL)}}$, by Lemma 1(c), and $\bar{F}_{X(1)}^{\text{DMRL(IMRL)}} \Rightarrow \bar{F}_{Y(n+1)|X(n)}^{\text{DMRL(IMRL)}}$ by Lemma 2. Thus we have

$$\bar{F}_{X(1)}^{\text{NBU(NWU)}} \Rightarrow \bar{F}_{Y(n+1)|X(n)}^{\text{DMRL(IMRL)}}.$$

To see that the above arguments lead us to a contradiction, we make use of Lemma 1(b) to establish that $\bar{F}_{X(1)}^{\text{IFRA(DFRA)}} \Rightarrow \bar{F}_{X(1)}^{\text{NBU(NWU)}} \Rightarrow \bar{F}_{Y(n+1)|X(n)}^{\text{IFR(DFR)}} \Rightarrow \bar{F}_{Y(n+1)|X(n)}^{\text{IFRA(DFRA)}}$, which according to Lemma 3 is false.

The proof of Theorem 2 is now complete.

Another counterexample which will immediately show nonclosure of the IFRA, NBU, and NBUE classes under minimal repair, is provided by the survival function of a parallel system of two independent components, each having an exponential survival function with scale parameter λ_i , $i = 1, 2$, and $\lambda_1 \neq \lambda_2$. However, this counterexample cannot be used to show nonclosure of the DFRA, NWU, and NWUE classes, and thus the need for Lemmas 3 and 4.

4. A BOUND FOR THE CONDITIONAL SURVIVAL FUNCTION

The preservation properties of Theorem 2 enable us to take advantage of the several known bounds and inequalities of reliability theory (Barlow and Proschan, 1975, p. 109) for

$\bar{F}_{Y(n+1)|X(n)}$, once we know its ageing characteristics. For example, bounds on $\bar{F}_{Y(n+1)|X(n)}$ can be obtained once we know that it is IFR (DFR). From Lemma 1, we also note that for all $n \in I$,

$$\bar{F}_{Y(n+1)|X(n)} \text{IFR(DFR)} \Rightarrow \bar{F}_{Y(n+1)|X(n)} \text{NBUE(NWUE)} \Rightarrow \bar{F}_{X(1)} \text{IFR(DFR)},$$

so that $\bar{F}_{Y(n+1)|X(n)}$ is IFR (DFR) if and only if $\bar{F}_{X(1)}$ is IFR (DFR).

However, the fact that $\bar{F}_{X(1)} \text{IFRA(DFRA)} \not\Rightarrow \bar{F}_{Y(n+1)|X(n)} \text{NBUE(NWUE)}$ (see Lemma 3) motivates us to develop some new bounds for

$\bar{F}_{Y(n+1)|X(n)}$. This is particularly germane since IFRA lifetimes can arise quite naturally in practice. For example, if we consider the coronary occlusion situation of Section 1, and treat each occlusion as a shock to the heart muscle, then from the theory of shock models and wear processes (Esary, Marshall, Proschan, 1973), it follows under some very general conditions that the time to failure of the heart muscle is IFRA. If a revival of the patient by CPR can be regarded as a minimal repair action, then by Lemma 3, the time to the second failure of the heart, conditional on the time to the first failure, cannot be described by any of the notions of ageing given in Definition 1. This motivates us to describe the life length $Y(n+1)|X(n)$ under the assumption of minimal repair, and under the assumption that $X(1)$ is IFRA, a task presently under investigation. A useful upper (lower) bound on $\bar{F}_{Y(n+1)|X(n)}$ motivated by the IFRA (DFRA) property of $\bar{F}_{X(1)}$ is given in Theorem 3.

Theorem 3. Let $\{X(n)\}$ satisfy (2.1), and let $\bar{F}_{X(1)}$ be IFRA (DFRA). Then, for all $n \in I$, $0 < y < \infty$ and $x \geq 0$,

$$\bar{F}_{Y(n+1)|X(n)}(y|x) \leq (\geq) [\bar{F}_{X(1)}(y+x)]^{(y)/(y+x)}.$$

Proof: $\bar{F}_{X(1)}$ IFRA(DFRA) \Rightarrow

$$\bar{F}_{X(1)}(\alpha x) \leq (\geq) [\bar{F}_{X(1)}(x)]^\alpha, \text{ for } 0 < \alpha < 1.$$

Thus

$$\bar{F}_{Y(n+1)|X(n)}(y|x) = \frac{\bar{F}_{X(1)}(y+x)}{\bar{F}_{X(1)}(x)} = \frac{\bar{F}_{X(1)}(y+x)}{\bar{F}_{X(1)}(\alpha(y+x))} = \frac{\bar{F}_{X(1)}(y+x)}{\bar{F}_{X(1)}(\alpha(y+x))}$$

for $\alpha = (x/(y+x))$. By the defining property of IFRA(DFRA),

$$\bar{F}_{Y(n+1)|X(n)}(y|x) \leq (\geq) \frac{\bar{F}_{X(1)}(y+x)}{[\bar{F}_{X(1)}(y+x)]^\alpha} = [\bar{F}_{X(1)}(y+x)]^{1-\alpha},$$

or that

$$\bar{F}_{Y(n+1)|X(n)}(y|x) \leq (\geq) [\bar{F}_{X(1)}(y+x)]^{y/(y+x)}.$$

5. COMPARISON OF THE STOCHASTIC PROCESSES GENERATED BY MINIMAL AND MAXIMAL REPAIR POLICIES

The ageing properties of a new item provide us with a vehicle for comparing the relative desirability of a minimal repair policy versus a maximal repair policy. This in turn helps us choose a particular maintenance policy. Specifically, for an item which ages, a maintenance policy involving a complete overhaul or replacement is more desirable than one involving minimal repair, provided that cost and resource considerations are put aside. The converse is true for items which improve with age. More formally, we have:

Theorem 4. If $\bar{F}_{X(1)}$ is NBU(NWU), then for all $i \in I$,

$$\begin{aligned} & P[Y(i+1) \geq t | X(i) = x \text{ and } \{X(n)\} \text{ satisfying (2.1)}] \\ & \leq (\geq) P[Y(i+1) \geq t | X(i) = x \text{ and } \{X(n)\} \text{ satisfying (2.2)}]. \end{aligned}$$

Proof: Obvious, since the left-hand side of the above is $\bar{F}_{X(1)}(t+x)/\bar{F}_{X(1)}(x)$, and the right-hand side of the above is $\bar{F}_{X(1)}(t)$.

In order to develop some properties of joint life lengths occurring under minimal and maximal repair policies, we shall first present a basic definition pertaining to a general relationship among independent and certain types of nonindependent life lengths.

Definition 3 (Barlow and Proschan, 1975, p. 29). Random variables $X = (X_1, \dots, X_n)$ are associated if

$$\text{Cov}[\Gamma(X), \Delta(X)] \geq 0$$

for all pairs of binary increasing functions Γ and Δ .

Association implies a positive dependence among random variables; in the context of reliability theory, this is often realistic. Conditions for the association of interfailure/restoration times $Y(i)$, $i = 1, \dots, n$ are given in Theorem 5 below.

Theorem 5. The sequence $\{Y(i)\}$, $i = 1, \dots, n$ is associated if

- (a) $\{X(n)\}$ satisfies (2.2)
- (b) $\{X(n)\}$ satisfies (2.1) and $\bar{F}_{X(1)}$ is DFR.

Proof: Part (a) follows trivially from the fact that when $\{X(n)\}$ satisfies (2.2), the $Y(i)$'s are independent, and independent random variables are associated. To prove (b), we can show that under the conditions of the theorem, $Y(1), \dots, Y(n)$ are conditionally nondecreasing in sequence (see Barlow and Proschan, 1975, p. 146), and are therefore associated.

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